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Extending the Bethe Ansatz: The Quantum Three-Particle Ring

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The quantum problem of three impenetrable particles of arbitrary mass confined to a ring is solved by the Bethe ansatz. The solution of this problem is intimately related to the solution a Helmholtz equation in the interior of an arbitrary acute triangle, a problem thought insoluble by Bethe ansatz methods.

KEY WORDS: Bethe ansatz; quantum chaos; quantum billiard; three-particle problem.

1. INTRODUCTION

This work is an outgrowth of an attempt to calculate the spectrum of the Laplace operator in the interior of an arbitrary triangle with Dirichlet boundary conditons on the sides. This is a problem of long standing[(1)] in which the Bethe ansatz was thought to fail, except for very special situations.[(2)]

After much frustration and effort we were led to examine a related problem, where, to our surprise, the Bethe ansatz succeeds. This problem (three impenetrable coordinates on a ring) is outlined below with the aim of describing it in the revolutionary language of *Baxterism* (i.e., star-triangle relations and commuting transfer matrices), as explained in the "Red Book of Quotations."[(3)] The relation between the two problems falls just short of being a one-to-one mapping, and is discussed in the last section.

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2. THREE COORDINATES ON A RING

2.1. Time Independent Schrödinger Equation

The coordinates x_1, x_2, x_3 label the particles of mass M_1, M_2, M_3. The time independent partial differential equation to be solved is

$$-\frac{\hbar^2}{2}\left(\frac{1}{M_1}\frac{\partial^2}{\partial x_1^2}+\frac{1}{M_2}\frac{\partial^2}{\partial x_2^2}+\frac{1}{M_3}\frac{\partial^2}{\partial x_3^2}\right)\Psi=E\Psi \tag{1}$$

Let $M_i = m_i M$ and

$$\left(\frac{1}{m_1}\frac{\partial^2}{\partial x_1^2}+\frac{1}{m_2}\frac{\partial^2}{\partial x_2^2}+\frac{1}{m_3}\frac{\partial^2}{\partial x_3^2}\right)\Psi=-k^2\Psi \tag{2}$$

where $k^2 = 2ME/\hbar^2$ and $m_1+m_2+m_3=1$.

We therefore seek the eigenvalues of an anisotropic Laplace operator in three dimensions. As a constraint of impenetrability we impose Dirichlet boundary conditions

$$\Psi(x_1, x, x)=\Psi(x, x_2, x)=\Psi(x, x, x_3)=0$$

The choice of Dirichlet boundary conditions is arbitrary. Any homogeneous boundary condition leads to zero flux through the boundary, and is thus impenetrable.

Geometrically the state function vanishes on planes in the 3-dimensional space where any pair of coordinates are equal, irrespective of the third coordinate. These planes intersect a center of mass plane where

$$m_1x_1+m_2x_2+m_3x_3=\text{constant}$$

In addition the coordinates are constrained by a ring condition

$$\Psi(x_1+nL, x_2+nL, x_3+nL)=\psi(x_{,1}, x_2, x_3)$$

i.e., if all three coordinates are translated around the ring the state function returns to its value.

2.2. Bethe Ansatz

We assume a form for the solution, a finite sum of plane waves, and show that this assumption is internally consistent. An individual plane wave is given by the amplitude

$$\phi=e^{\imath(k_1x_1+k_2x_2+k_3x_3)}=e^{i(k\cdot r)}$$

which is clearly a solution to the partial differential equation with eigenvalue

$$k^2 = \frac{k_1^2}{m_1} + \frac{k_2^2}{m_2} + \frac{k_3^2}{m_3} \tag{3}$$

The boundary conditions are not satisfied by a single plane wave, but the idea of the ansatz is that a properly chosen linear combination of plane waves will satisfy the boundary conditions. To that end consider a two-coordinate encounter, e.g., $x_1 = x_2 = x$. The linear combination of plane waves

$$\psi_0 = e^{\imath(k_1 x + k_2 x + k_3 x_3)} - e^{\imath(k_1' x + k_2' x + k_3 x_3)} \tag{4}$$

vanishes on this plane if

$$k_1 + k_2 = k_1' + k_2'$$

and has the same eigenvalue if

$$\frac{k_1^2}{m_1} + \frac{k_2^2}{m_2} = \frac{k_1'^2}{m_1} + \frac{k_2'^2}{m_2}$$

that is, if two-particle momentum and energy are conserved in the encounter. k_1', k_2', k_3' are linearly related to k_1, k_2, k_3 in an encounter of coordinates x_1 and x_2. A matrix representation of this linear relation is

$$\begin{pmatrix} k_1' \\ k_2' \\ k_3' \end{pmatrix} = \begin{pmatrix} \frac{m_1 - m_2}{m_1 + m_2} & \frac{2m_1}{m_1 + m_2} & 0 \\ \frac{2m_2}{m_1 + m_2} & \frac{-m_1 + m_2}{m_1 + m_2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}$$

or

$$k' = Ak$$

In encounters between other pairs of coordinates a similar set of linear relations arise. In an encounter $x_2 = x_3$,

$$k' = Bk$$

an encounter $x_1 = x_3$,

$$k' = Ck$$

where

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \dfrac{m_2 - m_3}{m_2 + m_3} & \dfrac{2m_2}{m_2 + m_3} \\ 0 & \dfrac{2m_3}{m_2 + m_3} & \dfrac{-m_2 + m_3}{m_2 + m_3} \end{pmatrix}$$

$$C = \begin{pmatrix} \dfrac{m_1 - m_3}{m_1 + m_3} & 0 & \dfrac{2m_1}{m_1 + m_3} \\ 0 & 1 & 0 \\ \dfrac{2m_3}{m_1 + m_3} & 0 & \dfrac{-m_1 + m_3}{m_1 + m_3} \end{pmatrix}$$

It is convenient to think of the primed amplitude as generated from the unprimed amplitude by an operator. For example, in the encounter between x_1 and x_2

$$\phi' = \mathscr{A}\phi = e^{\imath (Ak) \cdot r}$$

Of course,

$$\mathscr{A}\mathscr{B}\phi = e^{\imath (ABk) \cdot r}$$

The linear combination (4) is represented by

$$\psi_0 = (I - \mathscr{A})\, \phi$$

and

$$\mathscr{A}\psi_0 = -\psi_0$$

This linear combination of plane waves vanishes when $x_1 = x_2$. It does not vanish when any other pair of coordinates are equal. The ansatz is consistent if a linear combination of plane waves, Q, can be found such that

$$\mathscr{A}Q = \mathscr{B}Q = \mathscr{C}Q = -Q \tag{5}$$

2.3. Properties of the Matrices

The properties of the operators $\mathscr{A}, \mathscr{B}, \mathscr{C}$ are easily deduced from the corresponding properties of the matrices A, B, C. It is easily shown by direct multiplication that

$$A^2 = B^2 = C^2 = I \tag{6}$$

Unfortunately, no simple, accessible method is known to produce the fundamental property of the matrices which makes the factorization possible. If the reader has either great algebraic fortitude, or access to an algebraic computation program, it is easy to verify that

$$(ABC)^2 = I \tag{7}$$

Given the above identity (the analog of a star-triangle relation) it is easy to show that any product of any two matrices A, B, C, commutes with any other product of two (the analog of commuting transfer matrices).

Due to this commutation it is natural to choose a basis where products of two matrices are diagonal. To that end let

$$X_2 = AB, \qquad X_3 = BC, \qquad X_1 = CA$$

and

$$(X_i, X_j) = 0$$

Again, we offer no accessible proof, but the eigenvalues of the X_i are 1, $e^{\pm i\theta_i}$ where

$$\tan\frac{\theta_i}{2} = \sqrt{\frac{m_i}{m_j m_k}}$$

It is easily verified that

$$\tan\left(\frac{\theta_1}{2} + \frac{\theta_2}{2} + \frac{\theta_3}{2}\right) = 0$$

and therefore that the three angles $\theta_1/2$, $\theta_2/2$, $\theta_3/2$ are the angles of an acute triangle. It is algebraically and conceptually simpler to parameterize in terms of these angles. The masses are expressed as functions of these angles by

$$m_i = \frac{\tan(\theta_i/2)}{\tan(\theta_1/2) + \tan(\theta_2/2) + \tan(\theta_3/2)}$$

The angles of any triangle are arbitrarily close to rational multiples of π. Let the denominator of the rational multiple be a prime, P. The numerators N_1, N_2, N_3, of the rational multiple are a three-partition of P. That is

$$N_1 + N_2 + N_3 = P$$

Under this assumption

$$\frac{\theta_i}{2} = \frac{\pi N_i}{P}$$

and

$$X_1^P = X_2^P = X_3^P = 1$$

Under these conditions the sum

$$S = \sum_{s=1}^{P} (X_i)^s$$

is independent of i and satisfies

$$X_i S = S$$

or

$$ABS = BCS = CAS = S$$

from which it follows that

$$AS = BS = CS$$

The function which satisfies (5) may be variously written

$$Q = S - AS = S - BS = S - CS$$

Thus, the wavefunction is very complicated involving P terms which are powers of matrices. On the other hand the eigenvalue is quite simple, given by (3).

3. RING CONDITION

If all coordinates are uniformly translated around the ring the state function must return to its original value. Thus, for each plane wave amplitude

$$e^{\imath(k_1+k_2+k_3)L}=1$$

and thus

$$k_1+k_2+k_3=\frac{2n\pi}{L}$$

The additional condition in the relative coordinates is somewhat more complicated. As mentioned before the state function vanishes when any pair of coordinates are equal. This defines three planes in the coordinate space. These planes intersect in a line where $x_1=x_2=x_3$. On a ring this line is repeated periodically in the difference coordinates. These lines intersect the plane of fixed center of mass position in points. Figure 1 may be helpful in understanding the following discussion of the location of these points.

Suppose the coordinates are almost equal

$$x_1\approx x_2\approx x_3\approx 0$$

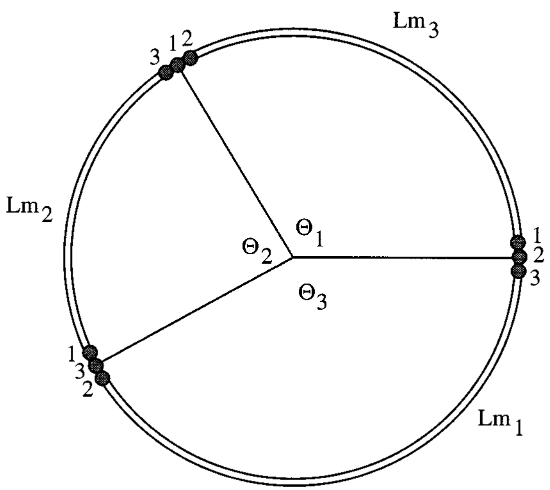

Fig. 1. Ring vertices for unequal masses.

and in clockwise order. Fix the spacing of x_1 and x_2 and move them together counterclockwise while moving x_3 clockwise keeping the center of mass position fixed. The three coordinates become almost equal at a second position where

$$(m_1+m_2)\,\Theta_1=m_3(L-\Theta_1)$$

that is, at a position

$$\Theta_1=m_3 L \tag{8}$$

with coordinates

$$x_1=x_2=m_3 L, \qquad x_3=-L(1-m_3)$$

Similarly, if the spacing of x_2 and x_3 is fixed and they are jointly moved clockwise as x_1 is moved counterclockwise the three coordinates again become almost equal at

$$\Theta_3=-m_1 L \tag{9}$$

with coordinates

$$x_1=L(1-m_1), \qquad x_2=x_3=-m_1 L$$

Each plane wave in the linear combination must take on the same value at each of these vertices

$$e^{\imath(k_1+k_2)\,\Theta_1}=e^{\imath k_3(L-\Theta_1)}$$

which implies

$$e^{\imath(k_1+k_2+k_3)\,m_3 L}=e^{\imath k_3 L}$$

Similar reasoning for the other vertices yield

$$e^{\imath(k_1 L/m_1)}=e^{\imath(k_2 L/m_2)}=e^{\imath(k_3 L/m_3)}=e^{2\imath\pi\Delta} \tag{10}$$

and therefore that

$$\frac{k_i L}{m_i}=2\pi(n_i+\Delta)$$

Periodicity under uniform translation requires

$$\sum k_i L = 2\pi \left(\sum (m_i n_i) + \Delta \right) = 2m\pi$$

thus,

$$\sum_1^3 m_i n_i + \Delta = m$$

Choosing $m = 0$, i.e., specializing to zero total momentum, implies

$$-\sum_1^3 m_i n_i = \Delta$$

the eigenvalue is

$$\sum_1^3 \frac{(k_i)^2}{m_i} = \frac{4\pi^2}{L^2} \sum_1^3 m_i (n_i + \Delta)^2 \tag{11}$$

Specializing to zero total momentum,

$$k^2 = \frac{4\pi^2}{L^2} \left(\sum_1^3 m_i n_i^2 - \left(\sum_1^3 m_i n_i \right)^2 \right) \tag{12}$$

4. ISOTROPIC SCHRÖDINGER EQUATION

If the coordinates of the Schrödinger equation, (2), are rescaled

$$y_i = g\sqrt{m_i}\, x_i$$

where g is a symmetric function of the m_i, the operator becomes a Laplace operator in three dimensions

$$\left(\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} + \frac{\partial^2}{\partial y_3^2} \right) \Psi = -g^2 k^2 \Psi \tag{13}$$

The eigenvalue is rescaled by g^2, but the dependence of the eigenvalue on the quantum numbers n_1, n_2, n_3 is unaffected by this transformation.

In the spherically symmetric y-space the distances between the vertices of the ring and the angles between the vectors which connect these vertices can be computed. The figure formed is a triangle with interior angles $\Theta_i/2$ and sides opposite those angles $gL\sqrt{m_i(1-m_i)}$. The eigenvalues may therefore be interpreted as the eigenvalue spectrum of a scalar Helmholtz equation in the interior of a triangle with Dirichlet boundary conditions on the sides. As of this writing, however, it is only possible to assert that the dependence of the spectrum on the quantum numbers n_1, n_2, n_3 is exact. The connection of the total mass of the particle system and the circumference of the confining ring to some linear dimension of this triangle remains a mystery.

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